Homework 3

This assignment is due by **11:59pm on Friday 9/20** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, take (nonblurry) photos/scans of your solutions, save them as a single pdf, and upload your file to gradescope (there are many free apps that allow you to scan and save documents as a pdf; e.g. camscanner). If you have not used gradescope before, see `gradescope.com/help`.

You may collaborate with your peers on homework assignments and to discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions. You must show all of your work to receive full credit.

1. Let P_{ij} be the permutation matrix that represents the row operation "Swap Row i and Row j" (i.e. multiplication of a matrix A on the left by P_{ij} swaps the ith and jth rows of A). Use the Gauss-Jordan algorithm to show that $P_{ij}^{-1} = P_{ij}$.

2. In class, we discussed elimination and permutation matrices. Multiplication on the left by an elimination matrix corresponds to a row replacement and multiplication on the left by a permutation matrix corresponds to swapping two rows. These are two of the three row operations needed to reduce a matrix to RREF. The final row operation is scaling.
   (a) Write down the 3×3 matrix S_2 that corresponds to multiplying the second row of a matrix by x (where $x \neq 0$). Verify your answer is correct by computing S_2A, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
   (b) More generally, describe the matrix S_i that corresponds to multiplying the ith row of a matrix by x (where $x \neq 0$).
   (c) Elimination matrices and permutation matrices are invertible. Is S_i invertible? If so, what is its inverse? Be sure to prove what you found is the inverse.

3. Determine which of the following sets are subspaces. If a set is a subspace, prove it. If a set is not a subspace, explain why.
   (a) $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \;\middle|\; xy = 0 \right\} \subset \mathbb{R}^2$.
   (b) $W = \left\{ \begin{bmatrix} 2x - y \\ y \end{bmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \subset \mathbb{R}^2$
   (c) $U = \left\{ \begin{bmatrix} x + y + z \\ 2x - y \\ 1 \end{bmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathbb{R}^3$

4. Let's consider intersections and unions of subspaces.
   (a)
   - Let L_1 and L_2 be two subspaces of $\mathbb{R}^2$ reprented by lines. What does $L_1 \cap L_2$ look like?
   - Let L and P be subspaces of $\mathbb{R}^3$ represented by a line and a plane, respectively. What does $L \cap P$ look like?
   - Let P_1 and P_2 be subspaces of $\mathbb{R}^3$ represented planes. What does $P_1 \cap P_2$ look like?

   (b) Let V and W be subspaces of $\mathbb{R}^n$. Prove that $V \cap W$ is also a subspace.

   (c) Let V and W be subspaces of $\mathbb{R}^n$. Show that $V \cup W$ is not necessarily a subspace by providing a counterexample.

5. True/False. If a statement is true, explain why. If a statement is false, explain why or provide a counterexample.
   (a) $N(A) = N(A^T)$.
   (b) The column space of $\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$ is a plane in $\mathbb{R}^3$.

6. Explain why there does not exist a matrix whose column space is a plane in $\mathbb{R}^3$ and whose nullspace is a line in $\mathbb{R}^4$.

7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$.
   (a) Use Julia to reduce A to RREF. Take a screenshot of your code and add it as its own page.

   (b) What are the pivot variables and free variables of the system $A\mathbf{x} = \mathbf{0}$?

   (c) Find the general solution to $A\mathbf{x} = \mathbf{0}$ and write it as a linear combination of special solutions.

   (d) Describe the nullspace of A as a span of vectors. What does the nullspace look like geometrically? In which vector space does it live?

   (e) Show that $\mathbf{x}_p = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$. Now use part (c) to find the general solution to $A\mathbf{x} = \mathbf{b}$.

8. Let A and B be any two matrices for which AB is defined. Prove that $C(AB)$ is a subset of $C(A)$.